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GENERALIZED STABILITY OF A GENERAL CUBIC FUNCTIONAL EQUATION

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Abstract. The general cubic functional equation is a generalization of many functional equations such as Jensen functional equation and the general quadratic functional equation. In this paper, we investigate the generalized stability of the general cubic functional equation

$$f(x+4y)-4f(x+3y)+6f(x+2y)-4f(x+y)+f(x)=0.$$

1. Introduction

In this paper, let V, X, and Y be a real vector space, a real normed space, and a real Banach space, respectively. To the problem of group isomorphism raised by Ulam [8] in 1940, the following year Hyers [2] provided a partial answer by solving the stability problem for the additive functional equation, and since then many mathematicians have generalized Hyers' stability results (see [1, 7] for more generalized results). Among them, the stability result shown by P. Găvruta below [1] can be said to be a major generalization of Hyers' results.

Proposition 1.1. *Let $(G,+)$ be an abelian group and $\varphi : G^2 \to [0,\infty)$ be a function such that*

$$\tilde{\varphi}(x,y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(x,y) < \infty$$

for all $x, y \in G$. If $f : G \to Y$ is a mapping such that

$$\|f(x+y)-f(x)-f(y)\| \le \varphi(x,y)$$

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for all $x, y \in G$, then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2}\tilde{\varphi}(x, x)$$

for all $x \in G$.

And then K. W. Jun et al. [3] and Y. H. Lee [5] have previously studied the stability of a general cubic functional equation

$$f(x + 4y) - 4f(x + 3y) + 6f(x + 2y) - 4f(x + y) + f(x) = 0 \qquad (1.1)$$

for all $x, y \in V$. If $f : V \to Y$ is a solution mapping of the functional equation (1.1), then we call the mapping f a general cubic mapping.

In this paper, we will give concise results that have improved the existing results on the stability of the general cubic functional equation through a clearer proof.

2. Stability of a general cubic functional equation

Throughout this paper, for a given mapping $f : V \to Y$, we use the following abbreviations:

$$\begin{aligned}
\tilde{f}(x) :=& f(x) - f(0),\\
f_1(x) :=& \frac{1}{12}\left(f(4x) - 12f(2x) + 32f(x)\right),\\
f_2(x) :=& -\frac{1}{8}\left(f(4x) - 10f(2x) + 16f(x)\right),\\
f_3(x) :=& \frac{1}{24}\left(f(4x) - 6f(2x) + 8f(x)\right),\\
\overset{4}{\underset{y}{\Delta}} f(x) :=& f(x + 4y) - 4f(x + 3y) + 6f(x + 2y) - 4f(x + y) + f(x),\\
\Gamma f(x) :=& f(8x) - 14f(4x) + 56f(2x) - 64f(x)
\end{aligned}$$

for all $x, y \in V$. By laborious computation we can get some useful equalities in the following lemma.

LEMMA 2.1. *For a given mapping $f : V \to Y$, the equalities*

$$f(x) = f_1(x) + f_2(x) + f_3(x) \tag{2.1}$$

$$\overset{4}{\underset{y}{\Delta}} \tilde{f}(x) = \overset{4}{\underset{y}{\Delta}} f(x), \tag{2.2}$$

$$\Gamma\tilde{f}(x) = \overset{4}{\underset{-2x}{\Delta}} f(8x) + 4 \overset{4}{\underset{x}{\Delta}} f(2x) + 16 \overset{4}{\underset{x}{\Delta}} f(x) + 20 \overset{4}{\underset{-x}{\Delta}} f(4x), \tag{2.3}$$

$$\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = -\frac{\Gamma\tilde{f}(x)}{24}, \quad \tilde{f}_1(x) - 2\tilde{f}_1\left(\frac{x}{2}\right) = \frac{1}{12}\Gamma\tilde{f}\left(\frac{x}{2}\right) \tag{2.4}$$

$$\tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = \frac{\Gamma\tilde{f}(x)}{32}, \quad \tilde{f}_2(x) - 4\tilde{f}\left(\frac{x}{2}\right) = -\frac{1}{8}\Gamma\tilde{f}\left(\frac{x}{2}\right) \tag{2.5}$$

$$\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = -\frac{\Gamma\tilde{f}(x)}{192}, \quad \tilde{f}_3(x) - 2\tilde{f}_3\left(\frac{x}{2}\right) = \frac{1}{24}\Gamma\tilde{f}\left(\frac{x}{2}\right) \tag{2.6}$$

hold for all $x, y \in V$.

LEMMA 2.2. *Let $f : V \to Y$ satisfy the functional equation*

$$\overset{4}{\underset{y}{\Delta}} f(x) = 0$$

for all $x, y \in V$, then we have

$$\tilde{f}_k(2x) = 2^k \tilde{f}_k(x) \tag{2.7}$$

for all $x \in V$ and each $k \in \{1, 2, 3\}$.

Proof. It is clear that $\Gamma\tilde{f}(x) = 0$ by (2.3). Therefore, the equality (2.7) follows from the equalities (2.4), (2.5), and (2.6). □

According to Corollary 6 in [4], we obtain following Lemma.

LEMMA 2.3. [4] *For a given mapping $f : V \to Y$, if there exist a mapping $F : V \to Y$ and a function $\phi : V \setminus \{0\} \to [0, \infty)$ that satisfy*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{2^i}\phi(2^i x) < \infty \quad or$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{2^{(\ell+1)i}}\phi(2^i x) + \sum_{i=0}^{\infty} 2^{\ell i}\phi\left(\frac{1}{2^i}x\right) < \infty \quad or$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} 2^{3i}\phi\left(\frac{1}{2^i}x\right) < \infty$$

for all $x \in V \setminus \{0\}$ *and for some* $\ell \in \{1, 2\}$*, where* $F(x) = \sum_{k=1}^{3} F_k(x)$ *and every* F_k *has the property (2.7), i.e.,* $F_k(2x) = 2^k F_k(x)$ *for all* $x \in V$*, then the mapping* F *is uniquely determined.*

LEMMA 2.4. *If a mapping* $f : V \to Y$ *satisfies the functional equation* $\overset{4}{\underset{y}{\Delta}} f(x) = 0$ *for all* $x, y \in V \setminus \{0\}$*, then it is a general cubic mapping.*

Proof. It is clear that $\overset{4}{\underset{0}{\Delta}} f(x) = 0$ for all $x \in V$ and

$$\overset{4}{\underset{y}{\Delta}} f(0) = \overset{4}{\underset{-y}{\Delta}} f(4y) = 0$$

for all $y \in V \setminus \{0\}$. So $\overset{4}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$ as desired. □

Now we show the theorem that improve the previous results(Theorem 3.2, Theorem 3.3, and Theorem 3.4 in [5]).

THEOREM 2.5. *Let* $\varphi : (V \setminus \{0\})^2 \to [0, \infty)$ *be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} 2^{-i} \varphi(2^i x, 2^i y) < \infty, \tag{2.8}$$

$$\sum_{i=0}^{\infty} 4^{-i} \varphi(2^i x, 2^i y) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.9}$$

$$\sum_{i=0}^{\infty} 8^{-i} \varphi(2^i x, 2^i y) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.10}$$

$$\sum_{i=0}^{\infty} 8^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.11}$$

for all $x, y \in V \setminus \{0\}$*. Suppose that* $f : V \to Y$ *is a mapping such that*

$$\left\| \overset{4}{\underset{y}{\Delta}} f(x) \right\| \leq \varphi(x, y) \tag{2.12}$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique general cubic mapping $F : V \to Y$ such that $F(0) = 0$ and

$$\|\tilde{f}(x) - F(x)\| \leq \frac{1}{24}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^i}, \tag{2.13}$$

$$\|\tilde{f}(x) - F(x)\| \leq \frac{1}{12}\sum_{i=0}^{\infty} 2^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{32}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{4^i}, \tag{2.14}$$

$$\|\tilde{f}(x) - F(x)\| \leq \frac{1}{8}\sum_{i=0}^{\infty} 4^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{192}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{8^i}, \tag{2.15}$$

$$\|\tilde{f}(x) - F(x)\| \leq \frac{1}{24}\sum_{i=2}^{\infty} 8^i \Phi\left(\frac{x}{2^{i+1}}\right), \tag{2.16}$$

for all $x \in V\backslash\{0\}$ if φ satisfies (2.8), (2.9), (2.10), or (2.11), respectively, where the mapping $\Phi : V\backslash\{0\} \to [0, \infty)$ is defined by

$$\Phi(x) := \varphi(8x, -2x) + 4\varphi(2x, x) + 16\varphi(x, x) + 20\varphi(4x, -x).$$

Proof. Notice that, from (2.3) and (2.12), we have

$$\begin{aligned} \|\Gamma\tilde{f}(x)\| &= \left\| \overset{4}{\underset{-2x}{\Delta}} f(8x) + 4 \overset{4}{\underset{x}{\Delta}} f(2x) + 16 \overset{4}{\underset{x}{\Delta}} f(x) + 20 \overset{4}{\underset{-x}{\Delta}} f(4x) \right\| \\ &\leq \Phi(x) \end{aligned} \tag{2.17}$$

for all $x, y \in V\backslash\{0\}$. We prove the theorem in two steps.

Step 1. Let $k \in \{1, 2, 3\}$ and $\delta \in \{-1, 1\}$, and let φ satisfy

$$\sum_{n=0}^{\infty} \frac{\varphi(2^{\delta n}x, 2^{\delta n}y)}{2^{\delta kn}} < \infty \tag{2.18}$$

for all $x, y \in V\backslash\{0\}$. From (2.4), (2.5), (2.6), and (2.17), we have the inequalities

$$\begin{aligned} \left\| \frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m} x)}{2^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1} x)}{2^{i+1}} \right) \right\| \\ &\leq \frac{1}{24} \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma\tilde{f}(2^i x)}{2^i} \right\| \leq \frac{1}{24} \sum_{i=n}^{n+m-1} \frac{\Phi(2^i x)}{2^i}, \end{aligned}$$

$$\begin{aligned}\left\|2^n\tilde{f}_1\left(\frac{x}{2^n}\right)-2^{n+m}\tilde{f}_1\left(\frac{x}{2^{n+m}}\right)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(2^i\tilde{f}_1\left(\frac{x}{2^i}\right)-2^{i+1}\tilde{f}_1\left(\frac{x}{2^{i+1}}\right)\right)\right\| \\ &\le \frac{1}{12}\sum_{i=n}^{n+m-1}\left\|2^i\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{12}\sum_{i=n}^{n+m-1}2^i\Phi\left(\frac{x}{2^{i+1}}\right),\end{aligned}$$

$$\begin{aligned}\left\|\frac{\tilde{f}_2(2^nx)}{4^n}-\frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_2(2^ix)}{4^i}-\frac{\tilde{f}_2(2^{i+1}x)}{4^{i+1}}\right)\right\| \\ &\le \frac{1}{32}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^ix)}{4^i}\right\| \le \frac{1}{32}\sum_{i=n}^{n+m-1}\frac{\Phi(2^ix)}{4^i},\end{aligned}$$

$$\begin{aligned}\left\|4^n\tilde{f}_2\left(\frac{x}{2^n}\right)-4^{n+m}\tilde{f}_2\left(\frac{x}{2^{n+m}}\right)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(4^i\tilde{f}_2\left(\frac{x}{2^i}\right)-4^{i+1}\tilde{f}_2\left(\frac{x}{2^{i+1}}\right)\right)\right\| \\ &\le \frac{1}{8}\sum_{i=n}^{n+m-1}\left\|4^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{8}\sum_{i=n}^{n+m-1}4^i\Phi\left(\frac{x}{2^{i+1}}\right),\end{aligned}$$

$$\begin{aligned}\left\|\frac{\tilde{f}_3(2^nx)}{8^n}-\frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_3(2^ix)}{8^i}-\frac{\tilde{f}_3(2^{i+1}x)}{8^{i+1}}\right)\right\| \\ &\le \frac{1}{192}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^ix)}{2^i}\right\| \le \frac{1}{192}\sum_{i=n}^{n+m-1}\frac{\Phi(2^ix)}{8^i},\end{aligned}$$

and

$$\begin{aligned}\left\|8^n\tilde{f}_3\left(\frac{x}{2^n}\right)-8^{n+m}\tilde{f}_3\left(\frac{x}{2^{n+m}}\right)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(8^i\tilde{f}_3\left(\frac{x}{2^i}\right)-8^{i+1}\tilde{f}_3\left(\frac{x}{2^{i+1}}\right)\right)\right\| \\ &\le \frac{1}{24}\sum_{i=n}^{n+m-1}\left\|8^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{24}\sum_{i=n}^{n+m-1}8^i\Phi\left(\frac{x}{2^{i+1}}\right)\end{aligned}$$

for all $x \in V\backslash\{0\}$ and $n, m \in \mathbb{N}\cup\{0\}$. It leads us to prove that $\left\{\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}}\right\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$ if φ satisfies (2.18). Moreover, since Y is complete and $\tilde{f}_k(0) = 0$, the sequence converges for all $x \in V$. It follows that we can define a mapping $F_{\delta k} : V \to Y$ by

$$F_{\delta k}(x) := \lim_{n\to\infty}\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}} \quad \text{if } \varphi \text{ satisfies (2.18).} \tag{2.19}$$

Now we observe that the equality

$$\begin{aligned}\overset{4}{\underset{y}{\Delta}} F_{\delta k}(x) &= F_{\delta k}(x+4y) - 4F_{\delta k}(x+3y) + 6F_{\delta k}(x+2y) \\ &\quad - 4F_{\delta k}(x+y) + F_{\delta k}(x) \\ &= \lim_{n\to\infty}\Big(\frac{\tilde f_k(2^{\delta n}(x+4y))}{2^{\delta n}} - 4\frac{\tilde f_k(2^{\delta n}(x+3y))}{2^{\delta n}} + 6\frac{\tilde f_k(2^{\delta n}(x+2y))}{2^{\delta n}} \\ &\quad - 4\frac{\tilde f_k(2^{\delta n}(x+y))}{2^{\delta n}} + \frac{\tilde f_k(2^{\delta n}x)}{2^{\delta n}}\Big)\end{aligned}$$

holds for all $x, y \in V\backslash\{0\}$. Together with the definition of $\tilde f_1$, if φ satisfies (2.18) for $k = 1$, then we have

$$\begin{aligned}\Big\|\overset{4}{\underset{y}{\Delta}} F_{\delta}(x)\Big\| &= \lim_{n\to\infty}\Big\|\frac{1}{12}\Big(\frac{\tilde f(2^{\delta n+2}(x+4y))}{2^{\delta n}} - 4\frac{\tilde f(2^{\delta n+2}(x+3y))}{2^{\delta n}}\Big) \\ &\quad + \frac{1}{12}\Big(6\frac{\tilde f(2^{\delta n+2}(x+2y))}{2^{\delta n}} - 4\frac{\tilde f(2^{\delta n+2}(x+y))}{2^{\delta n}} + \frac{\tilde f(2^{\delta n+2}x)}{2^{\delta n}}\Big) \\ &\quad - \Big(\frac{\tilde f(2^{\delta n+1}(x+4y))}{2^{\delta n}} - 4\frac{\tilde f(2^{\delta n+1}(x+3y))}{2^{\delta n}} + 6\frac{\tilde f(2^{\delta n+1}(x+2y))}{2^{\delta n}}\Big) \\ &\quad - \Big(-4\frac{\tilde f(2^{\delta n+1}(x+y))}{2^{\delta n}} + \frac{\tilde f(2^{\delta n+1}x)}{2^{\delta n}}\Big) \\ &\quad + \frac{8}{3}\Big(\frac{\tilde f(2^{\delta n}(x+4y))}{2^{\delta n}} - 4\frac{\tilde f(2^{\delta n}(x+3y))}{2^{\delta n}} + 6\frac{\tilde f(2^{\delta n}(x+2y))}{2^{\delta n}}\Big) \\ &\quad + \frac{8}{3}\Big(-4\frac{\tilde f(2^{\delta n}(x+y))}{2^{\delta n}} + \frac{\tilde f(2^{\delta n}x)}{2^{\delta n}}\Big)\Big\| \\ &= \lim_{n\to\infty}\Big\|\frac{\Delta^4_{2^{\delta n+2}y} f(2^{\delta n+2}x)}{12\cdot 2^{\delta n}} - \frac{\Delta^4_{2^{\delta n+1}y} f(2^{\delta n+1}x)}{2^{\delta n}} + \frac{8\,\Delta^4_{2^{\delta n}y} f(2^{\delta n}x)}{3\cdot 2^{\delta n}}\Big\| \\ &\le \lim_{n\to\infty}\Big(\frac{\varphi(2^{\delta n+2}x, 2^{\delta+2}y)}{12\cdot 2^{\delta n}} + \frac{\varphi(2^{\delta n+1}x, 2^{\delta n+1}y)}{2^{\delta n}} + \frac{8\varphi(2^{\delta n}x, 2^{\delta n}y)}{3\cdot 2^{\delta n}}\Big) \\ &= 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. In a similar way, by the definition of $\tilde f_2$, if φ satisfies (2.18) for $k = 2$, then we get

$$\Big\|\overset{4}{\underset{y}{\Delta}} F_{\delta 2}(x)\Big\| = \lim_{n\to\infty}\Big\| -\frac{1}{8}\Big(\frac{\tilde f(2^{\delta n+2}(x+4y))}{4^{\delta n}} - 4\frac{\tilde f(2^{\delta n+2}(x+3y))}{4^{\delta n}}\Big)$$

$$
\begin{aligned}
&-\frac{1}{8}\bigg(+6\frac{\tilde f(2^{\delta n+2}(x+2y))}{4^{\delta n}}-4\frac{\tilde f(2^{\delta n+2}(x+y))}{4^{\delta n}}+\frac{\tilde f(2^{\delta n+2}x)}{4^{\delta n}}\bigg)\\
&+\frac{5}{8}\bigg(\frac{\tilde f(2^{\delta n+1}(x+4y))}{4^{\delta n}}-4\frac{\tilde f(2^{\delta n+1}(x+3y))}{4^{\delta n}}+6\frac{\tilde f(2^{\delta n+1}(x+2y))}{4^{\delta n}}\bigg)\\
&+\frac{5}{8}\bigg(-4\frac{\tilde f(2^{\delta n+1}(x+y))}{4^{\delta n}}+\frac{\tilde f(2^{\delta n+1}x)}{4^{\delta n}}\bigg)\\
&-2\bigg(\frac{\tilde f(2^{\delta n}(x+4y))}{4^{\delta n}}-4\frac{\tilde f(2^{\delta n}(x+3y))}{4^{\delta n}}+6\frac{\tilde f(2^{\delta n}(x+2y))}{4^{\delta n}}\bigg)\\
&-2\bigg(-4\frac{\tilde f(2^{\delta n}(x+y))}{4^{\delta n}}+\frac{\tilde f(2^{\delta n}x)}{4^{\delta n}}\bigg)\bigg\|\\
=&\lim_{n\to\infty}\bigg\|\frac{\Delta^4_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{8\cdot 4^{\delta n}}-5\frac{\Delta^4_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{8\cdot 4^{\delta n}}+\frac{2\,\Delta^4_{2^{\delta n}y}f(2^{\delta n}x)}{4^{\delta n}}\bigg\|\\
\le&\lim_{n\to\infty}\bigg(\frac{\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{8\cdot 4^{\delta n}}+\frac{5\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{8\cdot 4^{\delta n}}+\frac{2\varphi(2^{\delta n}x,2^{\delta n}y)}{4^{\delta n}}\bigg)\\
=&\ 0
\end{aligned}
$$

for all $x, y \in V\backslash\{0\}$. Finally, by the definition of $\tilde f_3$, if φ satisfies (2.18) for $k = 3$, then we have

$$
\begin{aligned}
\bigg\|\overset{4}{\underset{y}{\Delta}}F_{\delta 3}(x)\bigg\| =&\lim_{n\to\infty}\bigg\|\frac{1}{24}\bigg(\frac{\tilde f(2^{\delta n+2}(x+4y))}{8^{\delta n}}-4\frac{\tilde f(2^{\delta n+2}(x+3y))}{8^{\delta n}}\bigg)\\
&+\frac{1}{24}\bigg(6\frac{\tilde f(2^{\delta n+2}(x+2y))}{8^{\delta n}}-4\frac{\tilde f(2^{\delta n+2}(x+y))}{8^{\delta n}}+\frac{\tilde f(2^{\delta n+2}x)}{8^{\delta n}}\bigg)\\
&-\frac{1}{4}\bigg(\frac{\tilde f(2^{\delta n+1}(x+4y))}{8^{\delta n}}-4\frac{\tilde f(2^{\delta n+1}(x+3y))}{8^{\delta n}}+6\frac{\tilde f(2^{\delta n+1}(x+2y))}{8^{\delta n}}\bigg)\\
&-\frac{1}{4}\bigg(-4\frac{\tilde f(2^{\delta n+1}(x+y))}{8^{\delta n}}+\frac{\tilde f(2^{\delta n+1}x)}{8^{\delta n}}\bigg)\\
&+\frac{1}{3}\bigg(\frac{\tilde f(2^{\delta n}(x+4y))}{8^{\delta n}}-4\frac{\tilde f(2^{\delta n}(x+3y))}{8^{\delta n}}+6\frac{\tilde f(2^{\delta n}(x+2y))}{8^{\delta n}}\bigg)\\
&+\frac{1}{3}\bigg(-4\frac{\tilde f(2^{\delta n}(x+y))}{8^{\delta n}}+\frac{\tilde f(2^{\delta n}x)}{8^{\delta n}}\bigg)\bigg\|\\
=&\lim_{n\to\infty}\bigg\|\frac{\Delta^4_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{24\cdot 8^{\delta n}}-\frac{\Delta^4_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{4\cdot 8^{\delta n}}+\frac{\Delta^4_{2^{\delta n}y}f(2^{\delta n}x)}{3\cdot 8^{\delta n}}\bigg\|
\end{aligned}
$$

$$\leq \lim_{n\to\infty} \left(\frac{\varphi(2^{\delta n+2}x, 2^{\delta n+2}y)}{24\cdot 8^{\delta n}} + \frac{\varphi(2^{\delta n+1}x, 2^{\delta n+1}y)}{4\cdot 8^{\delta n}} + \frac{\varphi(2^{\delta n}x, 2^{\delta n}y)}{3\cdot 8^{\delta n}} \right)$$
$$= 0$$

for all $x, y \in V\backslash\{0\}$. And then, since $\Delta_y^4 F_{\delta k}(x) = 0$ for all $x, y \in V\backslash\{0\}$, the mapping $F_{\delta k}$ is a general cubic mapping for all $k = 1, 2, 3$ and $\delta = \pm 1$ by Lemma 2.4.

Step 2. Now we define the desired general cubic mapping F for all cases.

(1) Let φ satisfy the condition (2.8), then F_1, F_2, and F_3 are well defined by (2.19). We put a general cubic mapping $F : V \to Y$ by

$$F(x) := F_1(x) + F_2(x) + F_3(x)$$

for all $x \in V$. Observe that.

$$\left\| \sum_{k=1}^{3} \tilde{f}_k(x) - \sum_{k=1}^{3} \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right\| \leq \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{3} \left(\frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}} \right) \right\|$$
$$\leq \sum_{i=0}^{n-1} \left(\frac{1}{24\cdot 2^i} - \frac{1}{32\cdot 4^i} + \frac{1}{192\cdot 8^i} \right) \left\| \Gamma\tilde{f}(2^i x) \right\|$$
$$\leq \sum_{i=0}^{n-1} \left\| \frac{\Gamma\tilde{f}(2^i x)}{24\cdot 2^i} \right\| \leq \frac{1}{24} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{2^i}$$

for all $x \in V\backslash\{0\}$, which follows (2.12) as $n \to \infty$.

(2) Let φ satisfy the condition (2.9), then F_{-1}, F_2, and F_3 are defined by (2.19). Putting a general cubic mapping $F : V \to Y$ by

$$F(x) := F_{-1}(x) + F_2(x) + F_3(x)$$

for all $x \in V$. Then we have

$$\left\| \tilde{f}_1(x) - 2^n \tilde{f}_1\left(\frac{x}{2^n}\right) - \sum_{k=2}^{3} \left(\tilde{f}_k(x) - \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right) \right\|$$
$$\leq \sum_{i=0}^{n-1} \left\| 2^i \tilde{f}_1\left(\frac{x}{2^i}\right) - 2^{i+1}\tilde{f}_1\left(\frac{x}{2^{i+1}}\right) \right\| + \sum_{i=0}^{n-1} \left\| \sum_{k=2}^{3} \left(\frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}} \right) \right\|$$

$$\le \sum_{i=0}^{n-1}\left\|\frac{2^i}{12}\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| + \sum_{i=0}^{n-1}\left(\frac{1}{32\cdot 4^i} - \frac{1}{192\cdot 8^i}\right)\left\|\Gamma\tilde{f}(2^i x)\right\|$$
$$\le \frac{1}{12}\sum_{i=0}^{n-1}2^i\Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{32}\sum_{i=0}^{n-1}\frac{\Phi(2^i x)}{4^i}$$

for all $x \in V\backslash\{0\}$, which follows (2.13) as $n \to \infty$.

(3) Let φ satisfy the condition (2.10), then F_{-1}, F_{-2}, and F_3 are defined by (2.19). Putting a general cubic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_3(x)$$

for all $x \in V$. We have the inequality

$$\left\|\sum_{k=1}^{2}\left(\tilde{f}_k(x) - 2^{kn}\tilde{f}_k\left(\frac{x}{2^n}\right)\right) + \tilde{f}_3(x) - \frac{\tilde{f}_3(2^n x)}{2^{3n}}\right\|$$
$$\le \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{2}\left(2^{ki}\tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)}\tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right)\right\|$$
$$+ \sum_{i=0}^{n-1}\left\|\frac{\tilde{f}_3(2^i x)}{2^{3i}} - \frac{\tilde{f}_3(2^{i+1}x)}{2^{3(i+1)}}\right\|$$
$$\le \sum_{i=0}^{n-1}\left(\frac{4^i}{8} - \frac{2^i}{12}\right)\left\|\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| + \sum_{i=0}^{n-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{192\cdot 8^i}\right\|$$
$$\le \frac{1}{8}\sum_{i=0}^{n-1}4^i\Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{192}\sum_{i=0}^{n-1}\frac{\Phi(2^i x)}{8^i}$$

for all $x \in V\backslash\{0\}$, which follows (2.14) as $n \to \infty$.

(4) Let φ satisfy the condition (2.11), then F_{-1}, F_{-2}, and F_{-3} are defined by (2.19). Putting a general cubic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x)$$

for all $x \in V$. We have the inequality

$$
\begin{aligned}
&\left\|\sum_{k=1}^{3}\left(\tilde{f}_k(x)-2^{kn}\tilde{f}_k\left(\frac{x}{2^n}\right)\right)\right\| \\
&\qquad\le \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{3}\left(2^{ki}\tilde{f}_k\left(\frac{x}{2^i}\right)-2^{k(i+1)}\tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right)\right\| \\
&\qquad\le \sum_{i=0}^{n-1}\left\|\left(\frac{2^i}{12}-\frac{4^i}{8}+\frac{8^i}{24}\right)\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\
&\qquad\le \sum_{i=2}^{n-1}\left\|\left(\frac{2^i}{12}-\frac{4^i}{8}+\frac{8^i}{24}\right)\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \;\le\; \frac{1}{24}\sum_{i=2}^{n-1}8^i\Phi\left(\frac{x}{2^{i+1}}\right)
\end{aligned}
$$

for all $x \in V\backslash\{0\}$, since $\frac{2^i}{12}-\frac{4^i}{8}+\frac{8^i}{24}=0$ when $i\in\{0,1\}$, which follows (2.15) as $n\to\infty$. Moreover, by the definition, we easily get

$$F_{\delta k}(2x)=2^kF_{\delta k}(x)$$

and $\overset{4}{\underset{y}{\Delta}} F_{\delta k}(x)=0$ for all $x,y\in V$. According to Lemma 2.4, F is the unique general cubic mapping. □

Using this theorem 2.5, we can obtain the following corollary that improve Corollary 3.6 in [5].

Corollary 2.6. [5] *Let θ be a positive real constant and p a real number such that $p\neq 1,2,3$. If $f: X\to Y$ satisfies the inequality*

$$\left\|\overset{4}{\underset{y}{\Delta}} f(x)\right\|\le\theta(\|x\|^p+\|y\|^p)$$

for all $x,y\in X\backslash\{0\}$, then there exists a mapping F such that $\overset{4}{\underset{y}{\Delta}} F(x)=0$ and

$$
\begin{aligned}
&\|\tilde{f}(x)-F(x)\|\le\frac{M\theta\|x\|^p}{12(2-2^p)} && \text{for}\quad p<1,\\
&\|\tilde{f}(x)-F(x)\|\le\frac{M\theta\|x\|^p}{12(2^p-2)}+\frac{M\theta\|x\|^p}{8(4-2^p)} && \text{for}\quad 1<p<2,\\
&\|\tilde{f}(x)-F(x)\|\le\frac{M\theta\|x\|^p}{8(2^p-4)}+\frac{M\theta\|x\|^p}{24(8-2^p)} && \text{for}\quad 2<p<3,\\
&\|\tilde{f}(x)-F(x)\|\le\frac{8M\theta\|x\|^p}{3\cdot 4^p(2^p-8)} && \text{for}\quad 3<p
\end{aligned}
$$

for all $x \in X\backslash\{0\}$, *where* $M := 8^p + 20 \cdot 4^p + 5 \cdot 2^p + 56$.

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